This expression has been discussed extensively (e.g., Ref. 4), and the interpretation of $d\chi/dt$ is that it describes the perturbative variation in the mean anomaly exclusive of the perturbations in the mean motion n. The form of $d\chi/dt$ as given by Eq. (4) of Ref. 1 is equivalent to the perturbative variation $\grave{M}$ of Herrick [cf. Eq. (190), Ref. 2]. Alternatively, it can be equated to the perturbative variation in the parameter $\epsilon - \tilde{\omega}$ of Plummer,[5] who also gives an expression for the total perturbative variation in the mean anomaly.

It follows from Eq. (1) and the definition of the mean longitude (i.e., $L = M + \omega + \Omega$) that

$$\frac{dL}{dt} = n + \frac{d\chi}{dt} + \frac{d\omega}{dt} + \frac{d\Omega}{dt} \tag{2}$$

where ω is the argument of the perifocus and Ω is the longitude of the ascending node. Therefore, the averaged motion in the mean longitude is given by

$$\overline{\frac{dL}{dt}} = \bar{n} + \overline{\frac{d\chi}{dt}} + \overline{\frac{d\omega}{dt}} + \overline{\frac{d\Omega}{dt}} \tag{3}$$

The only problem with this expression is in the interpretation of $\bar{n}$. The mean motion itself is given by an integral

$$n = n_0 + \int_{t_0}^{t} \left(\frac{dn}{dt}\right) dt \tag{4}$$

The average value of n with respect to the mean anomaly is, therefore, given by the following expression:

$$\bar{n} = n_0 + \frac{1}{2\pi} \int_0^{2\pi} \int_{t_0}^{t} \left(\frac{dn}{dt}\right) dt\, dM \tag{5}$$

The term to the right of the plus sign represents the deviation of the mean value of n from the initial value n_0 at the epoch t_0. Thus, if n_0 is used instead of $\bar{n}$ in the rate formula [Eq. (3)], than an additional term corresponding to the time integral of dn/dt enters in the expression for $\overline{dL/dt}$ or $\overline{dM/dt}$. For example, a formulation of this sort has been used by Edelbaum [Ref. 3, Eq. (21)].

Of course, when the periodic perturbations in L are required, it is necessary to include the extra term so that by Eqs. (2) and (4) the complete expression for the mean longitude is the following:

$$L - L_0 = n_0(t - t_0) + \int \int_{t_0}^{t} \left(\frac{dn}{dt}\right) dt^2 + \delta\chi + \delta\omega + \delta\Omega \tag{6}$$

where the symbol δ represents a variation from the unperturbed orbit with elements equal to the osculating elements at the epoch t_0.

Therefore,

$$\delta L = L - L_0 - n_0(t - t_0)$$

or, in agreement with Eq. (48) of Ref. 1, the perturbation in the mean longitude can be written in the following form:

$$\delta L = \int_{t_0}^{t} \delta n dt + \delta\chi + \delta\omega + \delta\Omega$$

Incidentally, we would like to take this opportunity to correct a typographical error in Eq. (46) of Ref. 1. That expression should read as follows:

$$\delta\omega = (\mu/2ac^2)[\tfrac{33}{2}\, e \sin M - (1 - \tfrac{23}{2}\, e^2) \sin 2M - \tfrac{3}{2}\, e \sin 3M - \tfrac{7}{4}\, e^2 \sin 4M]$$

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Comment on "Vibration of a 45° Right Triangular Cantilever Plate by a Gridwork Method"

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It is of interest that Christensen[1] considers a sophisticated version of the Hardy-Cross method to be better than the classical Rayleigh-Ritz procedure. The authors of this comment contend that an elaborated Rayleigh-Ritz method must inevitably give better results. This conviction is sustained by the proof that a first-order error in modal shape estimate causes only a second-order error in frequency. To realize the practical advantage of this useful theoretical result in a complex engineering structure, it is necessary to eliminate first-order errors in structural idealization. Ideally the elements must have the same deflections at common boundaries and also the same slopes; further discussion on structural idealization can be found in Refs. 2 and 3.

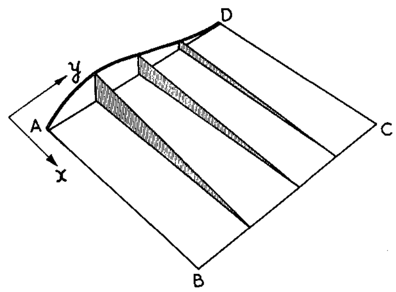

Fig. 1 Representation of displacement functions.

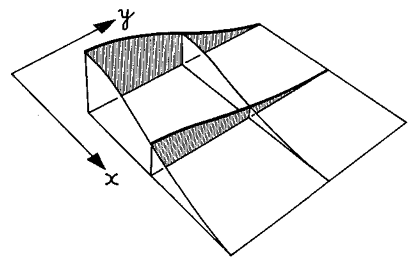

Fig. 2 Representation of displacement functions.

The results of a recent investigation by the authors, considering a 3- × 4-in. rectangular cantilevered plate of uniform thickness analyzed into 1-in.-square elements, have been included in this comment. The nodes were allowed three degrees of freedom in the out-of-plane directions, giving 48 degrees of freedom to the plate.

The element displacements assumed for unit deflections are typified by the following expressions, applicable to a rectangle with vertices $\pm a$, $\pm b$:

$$w = \tfrac{1}{8}\,\{x + a\}\,\{x^2/a^2 - 1\}\,\{y/b + 1\}$$

$$w = \tfrac{1}{16}\,\{-x^3/a^3 + 3x/a + 2\}\,\{-\,y^3/b^3 + 3y/b + 2\}$$

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Table 1 Frequencies for 3- × 4-in. plate

Mode	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Experimental frequency, cps	375	1142	2330	3909	4996	6580	8034	8099	12619	...	12819	14070	15443	18724	19931
Calculated frequency, cps	376	1197	2361	4083	4772	6796	8068	8427	12070	13063	13223	14428	15071	19219	20211
% error	0.27	2.5	1.3	4.2	−4.7	3.2	0.42	3.9	−4.5	...	3	2.5	−2.5	2.6	1.4

These displacement functions are illustrated in Figs. 1 and 2, and their choice gives rise to an inconsistent condition that one is obliged to accept.

In Fig. 1, the slope in the 0_y direction varies linearly along AB, i.e., the torsional strain is constant. This is desirable in the limiting case with small elements. Unfortunately, the displacement function fails to achieve this purpose for slopes in the 0_x direction, because at A and D the slope is zero, but along AD it is not; the same is true along BC. An alternative solution insures conformity, but at the expense of forcing zero torsion at every node. With the displacement function depicted by Fig. 2, this predicament does not arise.

The 12 × 12 stiffness matrix for the element was created by expressing the strain energy as a quadratic form in the nodal deflections and integrating algebraically. The kinetic energy, treated in similar fashion, yields an inertia matrix. For the final plate stiffness and inertia, the element inertias are assembled in exactly the same manner as the element stiffnesses. All the matrices are symmetrical.

Reference to Table 1 shows most of the calculated frequencies to be high; the three exceptions may, perhaps, be caused by the lack of slope conformity between elements.

Christensen probably owes his relative success to the introduction of cross masses, but the asymmetry of his inertia matrix is surprising. The inertia loads cannot be derivatives of a kinetic energy, and the equations of motion are not those of Lagrange. The first four equations of Ref. 4 provide a similarly based formulation that avoids this objection.

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